

# CERTAIN UNIFIED INTEGRATION FORMULAS ASSOCIATED WITH GENERALIZED $k$ -BESSEL FUNCTION

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ABSTRACT. Our purpose in this present paper is to investigate generalized integration formulas containing the generalized  $k$ -Bessel function  $W_{v,c}^k(z)$  to obtain the results in representation of Wright-type function. Also, we establish certain special cases of our main result.

## 1. INTRODUCTION

The generalized  $k$ -Bessel function defined in [11] as:

$$W_{v,c}^k(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left(\frac{z}{2}\right)^{2n + \frac{v}{k}}, \quad (1.1)$$

where  $k > 0$ ,  $v > -1$ , and  $c \in \mathbb{R}$  and  $\Gamma_k(z)$  is the  $k$ -gamma function defined in [5] as:

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt, \quad z \in \mathbb{C}. \quad (1.2)$$

By inspection the following relation holds:

$$\Gamma_k(z + k) = z\Gamma_k(z) \quad (1.3)$$

and

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right). \quad (1.4)$$

In the same paper, the researchers also defined Pochhammer  $k$ -symbols which is defined as:

$$(x)_{n,k} = x(x+k) \cdots (x+(n-1)k), \quad n \neq 0, n \in \mathbb{N}, (x)_{0,k} = 1.$$

The relation between Pochhammer  $k$ -symbols and  $k$ -gamma function is defined as:

$$(x)_{n,k} = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)}.$$

If  $k \rightarrow 1$  and  $c = 1$ , then the generalized  $k$ -Bessel function defined in (2.1) reduces to the well known classical Bessel function  $J_v$  defined in [7]. For further detail

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about  $k$ -Bessel function and its properties (see [8]-[10]).

The generalized hypergeometric function  ${}_pF_q(z)$  is defined in [6] as:

$${}_pF_q(z) = {}_pF_q \left[ \begin{matrix} (\alpha_1), (\alpha_2), \dots, (\alpha_p) \\ (\beta_1), (\beta_2), \dots, (\beta_q) \end{matrix} ; z \right] \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_q)_n} \frac{z^n}{n!}, \quad (1.5)$$

where  $\alpha_i, \beta_j \in \mathbb{C}$ ;  $i = 1, 2, \dots, p$ ,  $j = 1, 2, \dots, q$  and  $b_j \neq 0, -1, -2, \dots$  and  $(z)_n$  is the Pochhammer symbols. The gamma function is defined as:

$$\Gamma(\mu) = \int_0^{\infty} t^{\mu-1} e^{-t} dt, \mu \in \mathbb{C}, \quad (1.6)$$

$$\Gamma(z+n) = z\Gamma(z), z \in \mathbb{C}, \quad (1.7)$$

and beta function is defined as:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt. \quad (1.8)$$

The Wright type hypergeometric function is defined (see [16]-[18]) by the following series as:

$${}_p\Psi_q(z) = {}_p\Psi_q \left[ \begin{matrix} (\alpha_i, A_i)_{1,p} \\ (\beta_j, B_j)_{1,q} \end{matrix} ; z \right] \\ = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \dots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \dots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!} \quad (1.9)$$

where  $\beta_r$  and  $\mu_s$  are real positive numbers such that

$$1 + \sum_{s=1}^q \beta_s - \sum_{r=1}^p \alpha_r > 0.$$

Equation (3.1) differs from the generalized hypergeometric function  ${}_pF_q(z)$  defined (2.2) only by a constant multiplier. The generalized hypergeometric function  ${}_pF_q(z)$  is a special case of  ${}_p\Psi_q(z)$  for  $A_i = B_j = 1$ , where  $i = 1, 2, \dots, p$  and  $j = 1, 2, \dots, q$ :

$$\frac{1}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q \left[ \begin{matrix} (\alpha_1), \dots, (\alpha_p) \\ (\beta_1), \dots, (\beta_q) \end{matrix} ; z \right] = \frac{1}{\prod_{i=1}^p \Gamma(\alpha_i)} {}_p\Psi_q \left[ \begin{matrix} (\alpha_i, 1)_{1,p} \\ (\beta_j, 1)_{1,q} \end{matrix} ; z \right]. \quad (1.10)$$

In this paper, we define a class of integral formulas which containing the generalized  $k$ -Bessel function as defined in (1.1). Also, we investigate some special cases as the

corollaries. For this continuation of our study, we recall the following result of Lavoie and Trottier [12].

$$\int_0^1 z^{\alpha-1} (1-z)^{2\beta-1} \left(1 - \frac{z}{3}\right)^{2\alpha-1} \left(1 - \frac{z}{4}\right)^{\beta-1} dz = \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \Re(\alpha) > 0, \Re(\beta) > 0. \quad (1.11)$$

For various other investigation containing special function, the reader may refer to the recent work of researchers (see [3], [4], [13], [14], [15]).

## 2. Main Result

In this section, we establish two generalized integral formulas containing  $k$ -Bessel function defined (1.1), which represented in terms of Wright-type function defined in (1.9) by inserting with the suitable argument defined in (1.11).

**Theorem 2.1.** *For  $\lambda, \rho, v, c \in \mathbb{C}$  with  $\Re(\frac{v}{k}) > -1$ ,  $\Re(\lambda + \rho) > 0$ ,  $\Re(\lambda + \frac{v}{k}) > 0$  and  $z > 0$ , then the following result holds:*

$$\begin{aligned} & \int_0^1 z^{\lambda+\rho-1} (1-z)^{2\lambda-1} \left(1 - \frac{z}{3}\right)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{4}\right)^{\lambda-1} W_{v,c}^k \left( \frac{y \left(1 - \frac{z}{4}\right) \left(1 - z\right)^2}{2} \right) dz \\ &= \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda+\rho)}}{k^{\frac{v}{k}}} \\ & \quad \times {}_1\Psi_2 \left[ \begin{matrix} (\lambda + \frac{v}{k}, 2); \\ (\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \end{matrix} \middle| -\frac{cy^2}{4k} \right]. \quad (2.1) \end{aligned}$$

*Proof.* Let  $S$  be the left hand side of (2.1) and applying (1.1) to the integrand of (2.1), we have

$$\begin{aligned} S &= \int_0^1 z^{\lambda+\rho-1} (1-z)^{2\lambda-1} \left(1 - \frac{z}{3}\right)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{4}\right)^{\lambda-1} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left( \frac{y \left(1 - \frac{z}{4}\right) \left(1 - z\right)^2}{2} \right)^{2n + \frac{v}{k}} dz \end{aligned}$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.1, we have

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left(\frac{y}{2}\right)^{2n + \frac{v}{k}} \\ & \quad \times \int_0^1 z^{\lambda+\rho-1} (1-z)^{2(\lambda + \frac{v}{k} + 2n)-1} \left(1 - \frac{z}{3}\right)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{4}\right)^{\lambda + \frac{v}{k} + 2n-1} dz. \end{aligned}$$

By considering the assumption given in theorem 2.1, since  $\Re(\frac{v}{k}) > 0$ ,  $\Re(\lambda + \frac{v}{k} + 2n) > \Re(\lambda + \frac{v}{k}) > 0$ ,  $\Re(\lambda + \rho) > 0$ ,  $k > 0$  and applying (1.11), we obtain

$$S = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left(\frac{y}{2}\right)^{2n+\frac{v}{k}} \left(\frac{2}{3}\right)^{2(\lambda+\rho)} \frac{\Gamma(\lambda + \rho)\Gamma(\lambda + \frac{v}{k} + 2n)}{\Gamma(2\lambda + \rho + \frac{v}{k} + 2n)}.$$

Using (1.4), we get

$$S = \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda+\rho)}}{k^{\frac{v}{k}}} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\frac{v}{k} + 1 + n)n!} \left(\frac{y^{2n}}{4^n k^n}\right) \frac{\Gamma(\lambda + \frac{v}{k} + 2n)}{\Gamma(2\lambda + \rho + \frac{v}{k} + 2n)}$$

which upon using (1.9), we get the required result.  $\square$

**Theorem 2.2.** For  $\lambda, \rho, v, c \in \mathbb{C}$  with  $\Re(\frac{v}{k}) > -1$ ,  $\Re(\lambda + \rho) > 0$ ,  $\Re(\lambda + \frac{v}{k}) > 0$  and  $z > 0$ , then the following result holds:

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{3}\right)^{2\lambda-1} \left(1 - \frac{z}{4}\right)^{(\lambda+\rho)-1} W_{v,c}^k \left( \frac{yz \left(1 - \frac{z}{3}\right)^2}{2} \right) dz \\ &= \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda+\frac{v}{k})}}{k^{\frac{v}{k}}} \\ & \quad \times {}_1\Psi_2 \left[ \begin{matrix} (\lambda + \frac{v}{k}, 2); \\ (\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \end{matrix} \middle| -\frac{4cy^2}{81k} \right]. \quad (2.2) \end{aligned}$$

*Proof.* Let  $\mathfrak{L}$  be the left hand side of (2.2) and applying (1.1) to the integrand of (2.1), we have

$$\begin{aligned} \mathfrak{L} &= \int_0^1 z^{\lambda-1} (1-z)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{3}\right)^{2\lambda-1} \left(1 - \frac{z}{4}\right)^{(\lambda+\rho)-1} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left( \frac{yz \left(1 - \frac{z}{3}\right)^2}{2} \right)^{2n+\frac{v}{k}} dz \end{aligned}$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.2, we have

$$\begin{aligned} \mathfrak{L} &= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left(\frac{y}{2}\right)^{2n+\frac{v}{k}} \\ & \quad \times \int_0^1 z^{\lambda+\frac{v}{k}+2n-1} (1-z)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{3}\right)^{2(\lambda+\frac{v}{k}+2n)-1} \left(1 - \frac{z}{4}\right)^{\lambda+\rho-1} dz. \end{aligned}$$

By considering the assumption given in theorem 2.2, since  $\Re(\frac{v}{k}) > 0$ ,  $\Re(\lambda + \frac{v}{k} + 2n) > \Re(\lambda + \frac{v}{k}) > 0$ ,  $\Re(\lambda + \rho) > 0$ ,  $k > 0$  and applying (1.11), we obtain

$$\mathfrak{L} = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + k)n!} \left(\frac{y}{2}\right)^{2n+\frac{v}{k}} \left(\frac{2}{3}\right)^{2(\lambda+\frac{v}{k}+2n)} \frac{\Gamma(\lambda + \rho)\Gamma(\lambda + \frac{v}{k} + 2n)}{\Gamma(2\lambda + \rho + \frac{v}{k} + 2n)}.$$

Using (1.4), we get

$$S = \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda + \frac{v}{k} + 2n)}}{k^{\frac{v}{k}}} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\frac{v}{k} + 1 + n) n!} \left(\frac{y^{2n}}{4^n k^n}\right) \frac{\Gamma(\lambda + \frac{v}{k} + 2n)}{\Gamma(2\lambda + \rho + \frac{v}{k} + 2n)}$$

which upon using (1.9), we get the required result.  $\square$

### 3. Special Cases

In this section, we present the generalized form of classical and modified Bessel functions which are the special cases of  $k$ -Bessel function defined (1.1). Also, we prove two corollaries which are the special cases of obtained theorems in Section 2.

**Case 1.** If we set  $c = 1$  in (1.1), then we get another definition of  $k$ -Bessel function. We call it the classical  $k$ -Bessel function

$$J_v^k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\frac{v}{k} + 2n}}{\Gamma_k(v + nk + k)n!} \quad (3.1)$$

**Case 2.** If we set  $c = -1$  in (1.1), then we get another definition of  $k$ -Bessel function. We call it the modified  $k$ -Bessel function

$$I_v^k(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\frac{v}{k} + 2n}}{\Gamma(v + nk + k)n!} \quad (3.2)$$

**Corollary 3.1.** Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

$$\begin{aligned} & \int_0^1 z^{\lambda + \rho - 1} (1 - z)^{2\lambda - 1} \left(1 - \frac{z}{3}\right)^{2(\lambda + \rho) - 1} \left(1 - \frac{z}{4}\right)^{\lambda - 1} J_v^k \left( \frac{y \left(1 - \frac{z}{4}\right) \left(1 - z\right)^2}{2} \right) dz \\ &= \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda + \rho)}}{k^{\frac{v}{k}}} \\ & \quad \times {}_1\Psi_2 \left[ \begin{matrix} (\lambda + \frac{v}{k}, 2); \\ (\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \end{matrix} \middle| -\frac{y^2}{4k} \right]. \quad (3.3) \end{aligned}$$

**Corollary 3.2.** Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

$$\begin{aligned} & \int_0^1 z^{\lambda + \rho - 1} (1 - z)^{2\lambda - 1} \left(1 - \frac{z}{3}\right)^{2(\lambda + \rho) - 1} \left(1 - \frac{z}{4}\right)^{\lambda - 1} J_v^k \left( \frac{y \left(1 - \frac{z}{4}\right) \left(1 - z\right)^2}{2} \right) dz \\ &= \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda + \rho)}}{k^{\frac{v}{k}}} \\ & \quad \times {}_1\Psi_2 \left[ \begin{matrix} (\lambda + \frac{v}{k}, 2); \\ (\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \end{matrix} \middle| -\frac{y^2}{4k} \right]. \quad (3.4) \end{aligned}$$

**Corollary 3.3.** *Assume that the conditions of Theorem 2.2 are satisfied. Then the following integral formula holds:*

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{3}\right)^{2\lambda-1} \left(1 - \frac{z}{4}\right)^{(\lambda+\rho)-1} J_v^k \left( \frac{yz \left(1 - \frac{z}{3}\right)^2}{2} \right) dz \\ &= \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda + \frac{v}{k})}}{k^{\frac{v}{k}}} \\ & \quad \times {}_1\Psi_2 \left[ \begin{matrix} (\lambda + \frac{v}{k}, 2); \\ (\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \end{matrix} \middle| -\frac{4y^2}{81k} \right]. \quad (3.5) \end{aligned}$$

**Corollary 3.4.** *Assume that the conditions of Theorem 2.2 are satisfied. Then the following integral formula holds:*

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2(\lambda+\rho)-1} \left(1 - \frac{z}{3}\right)^{2\lambda-1} \left(1 - \frac{z}{4}\right)^{(\lambda+\rho)-1} I_v^k \left( \frac{yz \left(1 - \frac{z}{3}\right)^2}{2} \right) dz \\ &= \frac{\left(\frac{y}{2}\right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left(\frac{2}{3}\right)^{2(\lambda + \frac{v}{k})}}{k^{\frac{v}{k}}} \\ & \quad \times {}_1\Psi_2 \left[ \begin{matrix} (\lambda + \frac{v}{k}, 2); \\ (\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \end{matrix} \middle| -\frac{4y^2}{81k} \right]. \quad (3.6) \end{aligned}$$

**Remark.** *If we set  $k = 1$  in (3.1) to (3.6), then we get the well known result for case 1 (see [1]) and some new result for the familiar function defined in [11, 2, 19].*

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